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Short cuts to dynamic factor demand modelling

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Abstract

By means of so-called virtual or shadow prices, short-run factor demands, short-run marginal costs, etc. can be derived from any long-run cost function. The traditional approach (short-run/restricted/conditional/variable cost functions) is criticized, and it is also shown that technological change, scale effects, etc. can be added to any cost function by means of disembodied factor-augmenting efficiency indexes, easing the interpretations of the effects, but without loss of flexibility. It is shown that the trend- and scale-parameters of the (long-run) translog cost function can be directly translated into trend- and scale-parameters of such efficiency indexes. The techniques are illustrated on the well-known Berndt–Wood data set, using a (Diewert) long-run generalized Leontief (GL) cost function, and assuming that capital and labour are quasi-fixed. © 2000 Published by Elsevier Science S.A. All rights reserved.

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1. Introduction

Dynamic factor demand functions often originate from a postulated short-run (restricted/conditional/variable) cost function of the form $C = C(Y, X_1, P_1, \ldots,$

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P_n, t), where Y is the production level, X_1 is the level of the quasi-fixed factor (assuming here that there is only one such), P_1–P_n are the factor prices, and t is a time trend. As I see it, there are two problems with this approach. The first is that even if the mathematical form of $C(\cdot)$ may resemble and be inspired by some well-behaved long-run cost function such as, e.g. the translog or generalized Leontief, there is no guarantee that the corresponding long-run cost function implicitly contained in $C(\cdot)$ has the same well-behaved properties.[1] The second problem is that the way trend- and scale-effects (effects from t and Y) are incorporated into different cost functions is not standardized, making the effects difficult to interpret and compare across different cost functions, and making the cost functions themselves unnecessarily complex.

This paper proposes a general way of adding both dynamics, trend-, and scale-effects (or effects from other exogenous factors) to any long-run cost function in an easy and unambiguous manner. First, choose a no technical progress and constant returns to scale ("stripped down") long-run cost function, of the simple form $C^* = C^*(Y, P_1, \ldots, P_n)$, where $C^*(\cdot)$ is homogenous of degree one with respect to Y. Second, deduce all necessary short-run concepts (including the corresponding short-run cost function) from $C^*(\cdot)$, by means of so-called shadow or virtual prices of the quasi-fixed factors (cf. Sections 2.2, 3 and 5). Finally, add trend- and scale-effects to both the stripped down long-run cost function and its derived short-run counterpart by means of factor-specific factor-augmenting efficiency indexes of the form $e_i = e_i(t, Y)$ (cf. Section 6). Provided that the stripped down long-run cost function is flexible and well-behaved, these three steps yield a fully specified, flexible and well-behaved dynamic factor demand system, illustrated in the paper by a new and, in my view, superior way of deriving a dynamic factor demand system from the original long-run generalized Leontief cost function of Diewert (1971) (cf. Sections 2.2, 5–7).

Using the above-mentioned three-step approach, the effort can be concentrated on the first step; i.e. finding a suitable functional form for the stripped down long-run cost function $C^*(Y, P_1, \ldots, P_n) = Y \cdot c^*(P_1, \ldots, P_n)$. All the rest (dynamics, trend- and scale-effects) can be added in a fully mechanical, transparent and unambiguous way, by means of shadow prices and trend- and scale-dependent efficiency indexes. The trend- and scale-parameters of the efficiency indexes can be readily compared across different cost functions, and the trend-parameters of the efficiency indexes can be directly linked to the text book

[1] Regarding a short-run translog cost function, see e.g. Berndt et al. (1993), and regarding a short-run generalized Leontief cost function, see e.g. Morrison and Siegel (1997), building on Morrison (1988). Since the large body of knowledge concerning global curvature properties of the most commonly used long-run cost functions (see e.g. Caves and Christensen, 1980; Barnett and Lee, 1985; Despotakis, 1986; Diewert and Wales, 1987; Terrell, 1996; Perroni and Rutherford, 1998) does not apply to such postulated short-run cost functions, little is usually known of the global curvature properties of these.

concepts of Hicks-, Harrod-, and Solow-neutrality. The use of factor-augmenting efficiency indexes does not entail any loss of flexibility with respect to trend- and scale-effects, and it can be shown that all the trend- and scale-parameters of the well-known long-run translog cost function can be directly translated into trend- and scale-parameters of such efficiency indexes (cf. Section 6).

The paper has eight sections. In Sections 2.1 and 2.2, the traditional short-run cost function approach is presented, contrasting it with the proposed shadow price approach. Section 3 surveys the shadow price approach in the general n-factor case, and Section 4 discusses some theoretical problems with the traditional short-run cost function approach. Section 5 shows in the n-factor case how to form all relevant short-run concepts from the well-known long-run generalized Leontief cost function. Section 6 presents and discusses the efficiency index approach, and Section 7 gives a simple empirical application using the well-known Berndt–Wood data set. The conclusions are in Section 8.

2. Two different approaches to dynamic factor demand modelling

To provide a simple example, a particular cost function – the generalized Leontief of Diewert (1971) – is presented below in a short- and long-run version. For the sake of simplicity, it is assumed that there are only three production factors (one quasi-fixed, and two flexible), no technological change, and constant returns to scale.

2.1. The traditional short-run cost function approach

A widely used short-run generalized Leontief cost function originates from Morrison (1988) (also used in, e.g. Park and Kwon, 1995; Morrison and Siegel, 1997). This short-run version of the generalized Leontief is presented below, being fully representative of short-run cost functions in general, including the short-run translog. Subsequently, this short-run GL cost function is denoted the "traditional short-run generalized Leontief cost function", and with $n = 3$ production factors (one quasi-fixed and two flexible), no technological change, and constant returns to scale, total short-run costs, C, are given as[2]

$$C = P_K K + Y\left[\sum_i \sum_j \alpha_{ij}\, P_i^{0.5} P_j^{0.5} + \left(\frac{K}{Y}\right)^{0.5} (\delta_{KL} P_L + \delta_{KE} P_E) + \gamma_{KK}\left(\frac{K}{Y}\right)(P_L + P_E)\right], \tag{1}$$

[2] The function is taken from Morrison (1988), Eq. (1), with $\delta_{im} = \gamma_{mn} = \gamma_{mk} = 0$ (or Park and Kwon, 1995, Eq. (12) with the same restrictions). Please note that I have added the fixed costs, $P_K K$, on the right-hand side, so that (1) is the short-run *total* costs.

where $i, j = L, E$ and $\alpha_{LE} = \alpha_{EL}$. The variables K, L and E denote fixed capital, labour and energy, P_K, P_L and P_E are the corresponding factor prices, and Y is the production level. The first term of (1) is simply the fixed costs of K, whereas the second term is a generalized Leontief form in the prices of the two flexible factors (L and E). The last two terms express how the variable costs are affected by K, and the concrete mathematical form (1) contains $6 = n(n+1)/2$ parameters necessary for full flexibility (cf. Lau, 1974), but is otherwise quite arbitrary. Differentiating (1) with respect to P_L and P_E (Shephard's Lemma) yields the short-run factor demands:

$$L = Y\left[\alpha_{LL} + \alpha_{LE}\left(\frac{P_E}{P_L}\right)^{0.5} + \delta_{KL}\left(\frac{K}{Y}\right)^{0.5} + \gamma_{KK}\frac{K}{Y}\right], \tag{2}$$

$$E = Y\left[\alpha_{EE} + \alpha_{LE}\left(\frac{P_L}{P_E}\right)^{0.5} + \delta_{KE}\left(\frac{K}{Y}\right)^{0.5} + \gamma_{KK}\frac{K}{Y}\right]. \tag{3}$$

Short-run marginal costs, $MC \equiv \partial C/\partial Y$, are obtained by differentiating C with respect to Y:

$$MC = \sum_i \sum_j \alpha_{ij} P_i^{0.5} P_j^{0.5} + 0.5\left(\frac{K}{Y}\right)^{0.5}(\delta_{KL}P_L + \delta_{KE}P_E), \quad i, j = L, E. \tag{4}$$

At this point, a so-called shadow price of K, $\tilde{P}_K$, is introduced, denoting by how much short-run variable costs are reduced, if K is increased by one unit. Short-run variable costs are given as $G = P_L L + P_E E = C - P_K K$, implying $\tilde{P}_K = -\partial G/\partial K = P_K - \partial C/\partial K$, being in this case:

$$\tilde{P}_K = -0.5\left(\frac{K}{Y}\right)^{-0.5}(\delta_{KL}P_L + \delta_{KE}P_E) - \gamma_{KK}(P_L + P_E). \tag{5}$$

The long-run equilibrium condition implies that $\tilde{P}_K = P_K$ (equivalent to $\partial C/\partial K = 0$). This condition yields the long-run stock of capital, K^*, as

$$K^* = 0.25Y\left(\frac{-\delta_{KL}P_L - \delta_{KE}P_E}{P_K + \gamma_{KK}(P_L + P_E)}\right)^2. \tag{6}$$

Inserting K^* into the short-run demands for the flexible factors (L and E), yields the long-run demands, L^* and E^*:[3]

[3] The regularity conditions $\partial^2 C/\partial K^2 > 0$ and $\partial K^*/\partial P_K < 0$ together imply that $-\delta_{KL}P_L - \delta_{KE}P_E > 0$ and $P_K + \gamma_{KK}(P_L + P_E) > 0$ so that the numerator and denominator of the fractions are both positive. Therefore, no numerical sign is needed in the second fraction in (7) and (8), provided that the above conditions are observed. For a description of regularity conditions for short-run cost functions, see Browning (1983).

$$L^* = Y\left[\alpha_{LL} + \alpha_{LE}\left(\frac{P_E}{P_L}\right)^{0.5} + 0.50\,\delta_{KL}\left(\frac{-\delta_{KL}P_L - \delta_{KE}P_E}{P_K + \gamma_{KK}(P_L + P_E)}\right)\right.$$
$$\left. + 0.25\gamma_{KK}\left(\frac{-\delta_{KL}P_L - \delta_{KE}P_E}{P_K + \gamma_{KK}(P_L + P_E)}\right)^2\right], \quad (7)$$

$$E^* = Y\left[\alpha_{EE} + \alpha_{LE}\left(\frac{P_L}{P_E}\right)^{0.5} + 0.50\,\delta_{KE}\left(\frac{-\delta_{KL}P_L - \delta_{KE}P_E}{P_K + \gamma_{KK}(P_L + P_E)}\right)\right.$$
$$\left. + 0.25\gamma_{KK}\left(\frac{-\delta_{KL}P_L - \delta_{KE}P_E}{P_K + \gamma_{KK}(P_L + P_E)}\right)^2\right]. \quad (8)$$

If K^* is inserted into the short-run cost function (C), the corresponding long-run cost function (C^*) is obtained

$$C^* = Y\sum_i\sum_j \alpha_{ij}\,P_i^{0.5}\,P_j^{0.5} - 0.25\,Y\frac{(-\delta_{KL}P_L - \delta_{KE}P_E)^2}{P_K + \gamma_{KK}(P_L + P_E)}, \quad i,j = L, E. \quad (9)$$

Finally, long-run marginal costs, $MC^* \equiv \partial C^*/\partial Y$, are given as

$$MC^* = \sum_i\sum_j \alpha_{ij}\,P_i^{0.5}P_j^{0.5} - 0.25\frac{(-\delta_{KL}P_L - \delta_{KE}P_E)^2}{P_K + \gamma_{KK}(P_L + P_E)}, \; i,j = L, E. \quad (10)$$

2.2. The proposed shadow price approach

In this section, the procedure is turned upside down, beginning with a long-run cost function and ending up with a short-run cost function. Operating as previously with three production factors, no technological change, and constant returns to scale, the (original) long-run generalized Leontief cost function due to Diewert (1971) is

$$C^* = Y\sum_i\sum_j \beta_{ij}\,P_i^{0.5}\,P_j^{0.5}, \quad i,j = K, L, E, \quad \beta_{ij} = \beta_{ji}. \quad (11)$$

Long-run factor demands follow from Shephard's Lemma:

$$K^* = Y\,[\beta_{KK} + (\beta_{KL}P_L^{0.5} + \beta_{KE}P_E^{0.5})\,P_K^{-0.5}], \quad (12)$$

$$L^* = Y\,[\beta_{LL} + (\beta_{KL}P_K^{0.5} + \beta_{LE}P_E^{0.5})\,P_L^{-0.5}], \quad (13)$$

$$E^* = Y\,[\beta_{EE} + (\beta_{KE}P_K^{0.5} + \beta_{LE}P_L^{0.5})\,P_E^{-0.5}]. \quad (14)$$

Long-run marginal costs are obtained by differentiating (11) with respect to Y:

$$MC^* = \sum_i\sum_j \beta_{ij}\,P_i^{0.5}\,P_j^{0.5}, \quad i,j = K, L, E. \quad (15)$$

At this point, the question is how to get from the long-run cost function (or factor demands) to the corresponding system of short-run factor demands, where K is fixed at a predetermined level. The answer is to artificially alter the price of K, P_K, until K^* in (12) is equal to the predetermined level K. When the long-run demands for the flexible factors, L^* and E^*, are evaluated at this artificial price, they yield the short-run demands for those factors (see e.g. Neary and Roberts, 1980 or Squires, 1994 for proofs and details. See also Pollak, 1969; Deaton, 1986 regarding rationing and shadow prices).

In the literature on rationing in consumer demand systems, this price is usually denoted a "virtual" price following Rothbarth (1941), whereas it is more obvious to denote the artificial price a shadow price in the context of producer behavior. This is so, because it turns out that the shadow/virtual price concept defined here coincides with the shadow price concept defined in the preceding section; i.e. defined as $\tilde{P}_K = -\partial G/\partial K = P_K - \partial C/\partial K$ (see Squires, 1994, p. 238, for the proof). The shadow price of the quasi-fixed factor, $\tilde{P}_K$, is consequently found by solving K^* (12) with respect to its own price, P_K, yielding the following expression:

$$\tilde{P}_K = \left(\frac{\beta_{KL} P_L^{0.5} + \beta_{KE} P_E^{0.5}}{K/Y - \beta_{KK}} \right)^2. \tag{16}$$

Substituting $\tilde{P}_K$ for P_K in the long-run factor demands for the flexible factors ((13)–(14)), yields the short-run versions of those:

$$L = Y \left[\beta_{LL} + \left(\beta_{KL} \frac{\beta_{KL} P_L^{0.5} + \beta_{KE} P_E^{0.5}}{K/Y - \beta_{KK}} + \beta_{LE} P_E^{0.5} \right) P_L^{-0.5} \right], \tag{17}$$

$$E = Y \left[\beta_{EE} + \left(\beta_{KE} \frac{\beta_{KL} P_L^{0.5} + \beta_{KE} P_E^{0.5}}{K/Y - \beta_{KK}} + \beta_{LE} P_L^{0.5} \right) P_E^{-0.5} \right]. \tag{18}$$

The corresponding short-run cost function is given as $C = P_K K + P_L L + P_E E$, or

$$C = P_K K + Y \sum_i \sum_j \beta_{ij}\, P_i^{0.5} P_j^{0.5} + Y \frac{(\beta_{KL} P_L^{0.5} + \beta_{KE} P_E^{0.5})^2}{K/Y - \beta_{KK}}, \quad i, j = L, E \tag{19}$$

Regarding short-run marginal costs, the shadow price, $\tilde{P}_K$, can be used once again, as it can be shown that substituting $\tilde{P}_K$ for P_K in the long-run marginal costs, MC^*, yields the short-run marginal costs, MC (see Thomsen, 1998, Appendix A, for the proof). Alternatively, MC could be found by differentiating (19) with respect to Y. This yields, of course, the same.

$$MC = \sum_i \sum_j \beta_{ij}\, P_i^{0.5} P_j^{0.5} + (2K/Y - \beta_{KK}) \left(\frac{\beta_{KL} P_L^{0.5} + \beta_{KE} P_E^{0.5}}{K/Y - \beta_{KK}} \right)^2, \quad i.j = L, E. \tag{20}$$

Table 1
Overview of the shadow price approach (of Section 2.2)

	Long run	Short run
Demand for quasi-fixed factors	(a1) $\boldsymbol{X}_k^* = X_k^*(Y, \boldsymbol{P}_k, \boldsymbol{P}_l)$	(b1) $\boldsymbol{X}_k = X_k^*(Y, \tilde{\boldsymbol{P}}_k, \boldsymbol{P}_l)$
Demand for flexible factors	(a2) $\boldsymbol{X}_l^* = X_l^*(Y, \boldsymbol{P}_k, \boldsymbol{P}_l)$	(b2) $\boldsymbol{X}_l = X_l^*(Y, \tilde{\boldsymbol{P}}_k, \boldsymbol{P}_l)$
Marginal costs	(a3) $MC^* = MC^*(Y, \boldsymbol{P}_k, \boldsymbol{P}_l)$	(b3) $MC = MC^*(Y, \tilde{\boldsymbol{P}}_k, \boldsymbol{P}_l)$

Note: Exogenous variables: $Y, \boldsymbol{P}_k, \boldsymbol{P}_l$ and $\boldsymbol{X}_k$. Endogenous variables: $\boldsymbol{X}_k^*, \boldsymbol{X}_l^*, MC^*, \tilde{\boldsymbol{P}}_k, \boldsymbol{X}_l$ and MC.

3. Summarizing the proposed shadow price approach

Regarding the long-run cost functions of Sections 2.1 and 2.2, it is noted that the long-run version of the traditional short-run GL cost function (9), and the original Diewert long-run GL cost function (11) are different mathematical expressions with different properties. Actually, these equations do not have much in common, apart from the quadratic form of the prices of L and E. Comparing alternatively the short-run functions (Eqs. (1) and (19)), these are necessarily different as well.

In the following, the principles of the shadow price approach are summarized, this time in the general case with n factors, of which k are quasi-fixed and $l = n - k$ are flexible. The approach is illustrated in Table 1.

In the first column, the long-run factor demand functions and marginal cost function are shown. These are most often derived from a cost or production function. The variables are to be interpreted as follows: $\boldsymbol{X}_k$ is a vector of the k quasi-fixed factors, and $\boldsymbol{X}_l$ is a vector of the $l = n - k$ flexible factors. The variables $\boldsymbol{P}_k$ and $\boldsymbol{P}_l$ are vectors of the corresponding factor prices. Given the functional forms of $X_k^*(\cdot)$, $X_l^*(\cdot)$ and $MC^*(\cdot)$, the calculation of $\boldsymbol{X}_k^*, \boldsymbol{X}_l^*$ and MC^* is straightforward, since $Y, \boldsymbol{P}_k$ and $\boldsymbol{P}_l$ are exogenous variables.

Turning to the short-run behaviour, the first step is to find the k shadow prices, $\tilde{\boldsymbol{P}}_k$, ensuring that (b1) in Table 1 is observed – assuming here, that these shadow prices exist and are unique. With these k shadow prices, the calculation of $\boldsymbol{X}_l$ and MC is straightforward, since the functional forms, $X_l^*(\cdot)$ and $MC^*(\cdot)$ are reused. Short-run costs can be found by using the formula $C = C^*(Y, \tilde{\boldsymbol{P}}_k, \boldsymbol{P}_l) + (\boldsymbol{P}_k - \tilde{\boldsymbol{P}}_k)'\boldsymbol{X}_k$. This relationship is relatively straightforward, as $C^*(Y, \tilde{\boldsymbol{P}}_k, \boldsymbol{P}_l) = \tilde{\boldsymbol{P}}_k'\boldsymbol{X}_k + \boldsymbol{P}_l'\boldsymbol{X}_l$ (see e.g. Squires, 1994, p. 238). Alternatively, short-run costs can of course be computed as $C = \boldsymbol{P}_k'\boldsymbol{X}_k + \boldsymbol{P}_l'\boldsymbol{X}_l$, yielding the same. If needed, total short-run costs differentiated with respect to the k quasi-fixed factors are given as $\partial C/\partial \boldsymbol{X}_k = \boldsymbol{P}_k - \tilde{\boldsymbol{P}}_k$ (see Squires, 1994, p. 238).

If the shadow prices cannot be found analytically, they can be found by means of numerical methods. Alternatively, the appendix contains a convenient approximation formula, using long-run price elasticities to link short- and long-run factor demand.

4. Problems with the traditional short-run cost function approach

Apart from being less convenient and transparent (in my view), the traditional short-run cost function approach tends to suffer from unrealistic and asymmetric isoquants of the underlying production function. For instance, this section shows that the traditional short-run GL cost function of Section 2.1 does not fully live up to its name (containing the Leontief case as a special case), since very peculiar isoquants result from driving the substitution between the quasi-fixed factor and the other factors close to zero.

The problem is illustrated by comparing the traditional short-run GL cost function of Section 2.1 with the Diewert long-run GL cost function of Section 2.2. The six parameters of each of the two cost functions are chosen so that they both yield $K^* = L^* = E^* = 1$ at prices $P_K = P_L = P_E = 1$ and production level $Y = 1$. In addition, the parameters are chosen so that the long-run partial price elasticities of the two functions are equal in the considered point, as all own-price partial price elasticities are set equal to -0.33, and all cross-price partial elasticities are set equal to 0.17 (i.e. halfway between the Leontief and the Cobb–Douglas special cases).[4]

By altering the factor prices, the underlying isoquant may be depicted using the long-run factor demand functions ((6)–(8) and (12)–(14), respectively). In Figs. 1 and 2, origo is in the most distant bottom right-hand corner, and three lines intersect at the central point, (1, 1, 1), showing how the factor demand system responds to modifications in one of the three factor prices, respectively.[5]

In Fig. 1, the line from the center of the figure in the direction of the black square indicates how the three factors respond to a reduction in the price of K. This increases the demand for K and decreases the demand for both L and E. The main problem of the traditional short-run GL cost function is that when the price of K is diminished towards zero, the demand for all three factors tends

[4] In the considered "point", all factors are net substitutes, and all cost shares are equal. Own-price partial price elasticities of -0.33 are not all implausible, and elasticities of that magnitude can, e.g. be found in Morrison (1988), or in Section 7 of this paper. Here, the parameters in the traditional GL are $\alpha_{LL} = \alpha_{EE} = 4$, $\alpha_{LE} = 0.5$, $\delta_{KL} = \delta_{KE} = -6$ and $\gamma_{KK} = 2.5$. In the Diewert GL all the β_{ij}'s are equal to $\frac{1}{3}$.

[5] Technical note: the circuits in the 3-D graphs are constructed by making the three factor prices run through the values $\ln(P_i) = \frac{2}{3}\ln(R)\sin(x + \frac{2}{3}\pi(i-1))$, $i = 1, 2, 3$ and $0 \leqslant x < 2\pi$, R being 1.20, 1.44, …, 5.16, respectively. Hence, the circuits depict twists in the relative factor prices of 20%, and 44%, etc., up to a factor 5.16.

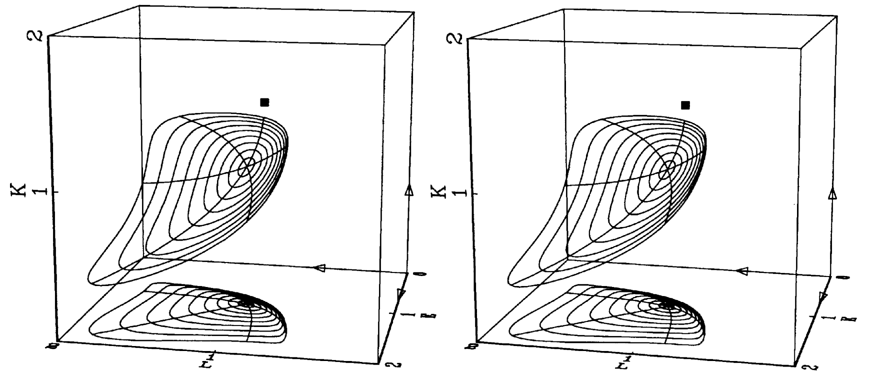

Fig. 1. Isoquant for the traditional short-run GL cost function.

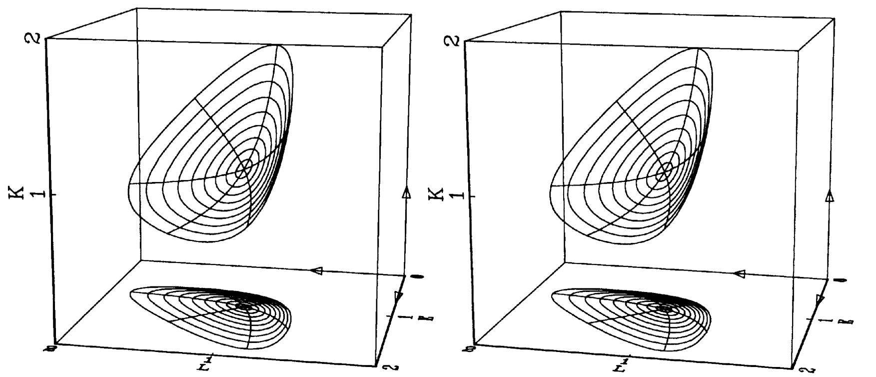

Fig. 2. Isoquant for the Diewert long-run GL cost function. *Note*: Figs. 1 and 2 are two paired stereograms, so that a 3-D image may be obtained, when the left and the right images are made to overlap by the eyes. The pictures at the bottom of each box are 2-D "shadows" of the above figures.

towards a particular positive level (marked by the black square), in contrast to the Diewert long-run GL cost function (see Fig. 2), where K tends towards infinity.[6] Increasing P_K in the traditional short-run GL cost function in Fig. 1 means following the above-mentioned line in the reverse direction, and it is seen that increasing P_K by a factor 5.16 (the outermost circuit) gives dramatic

[6] In Eq. (6), K^* tends towards a finite positive level, when P_K tends towards zero (δ_{KL} and δ_{KE} are both negative, and γ_{KK} is positive). In (12) K^* tends towards infinity when P_K tends towards zero (all β's are positive).

increases in the demand for L and E, compared to the Diewert long-run GL cost function.[7]

What is particularly unpleasant about this is that the traditional short-run GL cost function could not be rejected on the grounds of being theoretically inconsistent with the neoclassical assumptions. In fact, the isoquant in Fig. 1 (and 2 as well) is globally convex and does not yield negative factor demands anywhere. This means that the concept of "well-behavedness" or global consistency could perhaps need to be tightened, since it does not, e.g. rule out isoquants implying that all demands converge towards a specific strictly positive level (the black square in Fig. 1), when a chosen factor price (here: P_K) is driven towards zero. With the traditional short-run GL cost function, this problem becomes worse, the smaller the substitution between K and the other factors. In this respect, the traditional short-run GL does not fully live up to its name (containing the Leontief case as a special case).[8]

5. Using the (Diewert) long-run GL cost function generally

In this section, the results concerning the (Diewert) long-run generalized Leontief cost function of Section 2.2 are generalized to the n factor case with k quasi-fixed factors and $l = n - k$ flexible factors. However, this section could also be regarded as a specific application of the general long-run cost function framework presented in Section 3.

The GL cost function is extended slightly compared to Section 2.2, as there are no longer constant returns to scale, but instead – as in Diewert's original paper – the underlying production function is homothetic. Thus, Y in (11) is replaced by $h(Y)$, where it is assumed that $h(0) = 0$, $h'(Y) > 0$, and $h(Y)$ tends towards infinity as Y tends towards infinity. The homothetic GL cost function is given as (see Diewert, 1971)

$$C^* = h(Y)\,\boldsymbol{P}^{0.5'}\,\boldsymbol{B}\,\boldsymbol{P}^{0.5}, \tag{21}$$

[7] Of course, nobody says that the isoquant of the Diewert long-run GL cost function is the truth, but actually this isoquant and a three-factor CES-isoquant with elasticity of substitution $\sigma = 0.50$ and equal cost shares turn out to be identical. Therefore, the conclusion is that the traditional short-run GL cost function differs much from a globally well-behaved functional form such as the CES – at least in this (not unreasonable) case given the chosen elasticities and cost shares.

[8] Regarding the traditional formulation of the short-run translog cost function (for the first use of this form, see Atkinson and Halvorsen, 1976; for recent examples, see e.g. Shah, 1992; Berndt et al., 1993; Nemoto et al., 1993), this suffers from exactly the same problem. Such a short-run translog – where K is introduced in the quadratic form in the same way as the factor prices – has the further disadvantage that it is not possible to solve the equation yielding K^* analytically (in closed form).

where $\boldsymbol{P}$ is a $n\times 1$ column vector of the n factor prices, $\boldsymbol{B}=[\beta_{ij}]$ is a $n\times n$ symmetric matrix of parameters, and where the square root symbol means that the square root of each element is taken. Shephard's Lemma, $\boldsymbol{X}^{*}=\partial C^{*}/\partial \boldsymbol{P}$, yields the long-run factor demands:

$$\boldsymbol{X}^{*}=h(Y)\,\hat{\boldsymbol{P}}^{-0.5}\,\boldsymbol{B}\,\boldsymbol{P}^{0.5}, \tag{22}$$

where $\boldsymbol{X}$ is a $n\times 1$ column vector of factor levels, and where $\hat{\boldsymbol{P}}$ denotes the diagonalization of $\boldsymbol{P}$ into a $n\times n$ diagonal matrix. The n production factors are now divided into k quasi-fixed factors and $l=n-k$ flexible factors, so that (22) is partitioned into

$$\begin{bmatrix}\boldsymbol{X}_k^{*}\\ \boldsymbol{X}_l^{*}\end{bmatrix}=h(Y)\begin{bmatrix}\hat{\boldsymbol{P}}_k^{-0.5} & \boldsymbol{0}\\ \boldsymbol{0} & \hat{\boldsymbol{P}}_l^{-0.5}\end{bmatrix}\begin{bmatrix}\boldsymbol{B}_{kk} & \boldsymbol{B}_{kl}\\ \boldsymbol{B}'_{kl} & \boldsymbol{B}_{ll}\end{bmatrix}\begin{bmatrix}\boldsymbol{P}_k^{0.5}\\ \boldsymbol{P}_l^{0.5}\end{bmatrix},\quad \boldsymbol{B}_{kk}=\boldsymbol{B}'_{kk},\boldsymbol{B}_{ll}=\boldsymbol{B}'_{ll}. \tag{23}$$

That is,

$$\boldsymbol{X}_k^{*}=h(Y)\,\hat{\boldsymbol{P}}_k^{-0.5}\,(\boldsymbol{B}_{kk}\boldsymbol{P}_k^{0.5}+\boldsymbol{B}_{kl}\boldsymbol{P}_l^{0.5}), \tag{24}$$

$$\boldsymbol{X}_l^{*}=h(Y)\,\hat{\boldsymbol{P}}_l^{-0.5}\,(\boldsymbol{B}'_{kl}\boldsymbol{P}_k^{0.5}+\boldsymbol{B}_{ll}\boldsymbol{P}_l^{0.5}). \tag{25}$$

The next step is to isolate $\boldsymbol{P}_k$ from (24) to obtain the shadow prices of the quasi-fixed factors:

$$\tilde{\boldsymbol{P}}_k=[(\hat{\boldsymbol{X}}_k/h(Y)-\boldsymbol{B}_{kk})^{-1}\boldsymbol{B}_{kl}\boldsymbol{P}_l^{0.5}]^{2}, \tag{26}$$

provided that the matrix $(\hat{\boldsymbol{X}}_k/h(Y)-\boldsymbol{B}_{kk})$ is non-singular. The square symbol means squaring each element of the vector, and the derivation exploits $\hat{\boldsymbol{P}}_k^{0.5}\boldsymbol{X}_k=\hat{\boldsymbol{X}}_k\boldsymbol{P}_k^{0.5}$. The matrix $(\hat{\boldsymbol{X}}_k/h(Y)-\boldsymbol{B}_{kk})$ could be called the "characteristic" matrix, as its inverted counterpart, $(\hat{\boldsymbol{X}}_k/h(Y)-\boldsymbol{B}_{kk})^{-1}$, describes how the levels of the quasi-fixed factors affect the flexible factors in the short run. Inserting $\tilde{\boldsymbol{P}}_k$ in the place of $\boldsymbol{P}_k$ in the long-run demand equations for the flexible factors (25) yields the short-run demands for the same

$$\boldsymbol{X}_l=h(Y)\,\hat{\boldsymbol{P}}_l^{-0.5}\,[\boldsymbol{B}'_{kl}(\hat{\boldsymbol{X}}_k/h(Y)-\boldsymbol{B}_{kk})^{-1}\boldsymbol{B}_{kl}+\boldsymbol{B}_{ll}]\,\boldsymbol{P}_l^{0.5}. \tag{27}$$

Short-run costs are most easily found by using $C=\boldsymbol{P}'_k\boldsymbol{X}_k+\boldsymbol{P}'_l\boldsymbol{X}_l$, yielding

$$C=\boldsymbol{P}'_k\boldsymbol{X}_k+h(Y)\boldsymbol{P}_l^{0.5\prime}\,\boldsymbol{B}_{ll}\boldsymbol{P}_l^{0.5}+h(Y)\boldsymbol{P}_l^{0.5\prime}\boldsymbol{B}'_{kl}(\hat{\boldsymbol{X}}_k/h(Y)-\boldsymbol{B}_{kk})^{-1}\boldsymbol{B}_{kl}\boldsymbol{P}_l^{0.5}. \tag{28}$$

Long-run marginal costs are found by differentiating (21) with respect to Y:

$$MC^{*}=h'(Y)\boldsymbol{P}^{0.5\prime}\,\boldsymbol{B}\,\boldsymbol{P}^{0.5}. \tag{29}$$

Partitioning again into the quasi-fixed and flexible factors, yields

$$MC^{*}=h'(Y)\,[\boldsymbol{P}_k^{0.5}\;\boldsymbol{P}_l^{0.5}]\begin{bmatrix}\boldsymbol{B}_{kk} & \boldsymbol{B}_{kl}\\ \boldsymbol{B}'_{kl} & \boldsymbol{B}_{ll}\end{bmatrix}\begin{bmatrix}\boldsymbol{P}_k^{0.5}\\ \boldsymbol{P}_l^{0.5}\end{bmatrix}. \tag{30}$$

Inserting the k shadow prices into MC^* gives the short-run marginal costs, MC, as

$$MC = h'(Y)\,[\tilde{\boldsymbol{P}}_k^{0.5\prime}\boldsymbol{B}_{kk}\tilde{\boldsymbol{P}}_k^{0.5} + 2\tilde{\boldsymbol{P}}_k^{0.5\prime}\boldsymbol{B}_{kl}\boldsymbol{P}_l^{0.5} + \boldsymbol{P}_l^{0.5\prime}\boldsymbol{B}_{ll}\boldsymbol{P}_l^{0.5}]. \tag{31}$$

If the shadow prices, $\tilde{\boldsymbol{P}}_k$ (26), are substituted into (31), MC is given as a function of $Y, \boldsymbol{P}_l$ and $\boldsymbol{X}_k$.

Finally – addressing the underlying production function – as it has been shown on the previous pages, it is not at all necessary to know the functional form of the underlying production function, but, as it is quite simple to deduce an expression yielding it, it is derived below. The procedure is to assume that $n-1$ out of the n factors are fixed and subsequently derive the short-run demand for the nth factor. Assuming that there is only one flexible factor in (27), $\hat{\boldsymbol{P}}_l^{-0.5}$ and $\boldsymbol{P}_l^{0.5}$ are both scalars and cancel out, yielding

$$\boldsymbol{X}_l = h(Y)\,[\boldsymbol{B}_{kl}'(\hat{\boldsymbol{X}}_k/h(Y) - \boldsymbol{B}_{kk})^{-1}\boldsymbol{B}_{kl} + \boldsymbol{B}_{ll}]. \tag{32}$$

Here, $\boldsymbol{X}_l$ and $\boldsymbol{B}_{ll}$ are scalars, and $\boldsymbol{B}_{kl}$ is a $(n-1)\times 1$ column vector. This equation gives a relationship between the n production factors, X_1–X_n, and the production level, Y – i.e. the underlying production function. It can be shown that (32) is equivalent to

$$\begin{vmatrix} \hat{\boldsymbol{X}}_k/h(Y) - \boldsymbol{B}_{kk} & -\boldsymbol{B}_{kl} \\ -\boldsymbol{B}_{kl}' & \boldsymbol{X}_l/h(Y) - \boldsymbol{B}_{ll} \end{vmatrix} = 0 \tag{33}$$

or

$$|\hat{\boldsymbol{X}}/h(Y) - \boldsymbol{B}| = 0. \tag{34}$$

Therefore, the underlying dual "generalized Leontief production function" is given by the condition that the "full" characteristic matrix is singular; i.e. that this matrix has zero determinant. Generally, Y would be given as the solution to a polynomial of degree n, so it should be stressed that (34) only gives *necessary* conditions for the production function.

6. Efficiency indexes, trend- and scale-effects, etc.

Up to now, we have abstracted almost completely from trend- and scale-effects (or effects from other exogenous factors). This has been intentional, as such generalizations can be implemented in a very convenient and unambiguous way, provided that one is willing to accept the simplifying (but not restricting, see below) assumption that the effects of technological change (t) and the production level (Y) are purely *factor-augmenting*; i.e. affecting each factor through a factor-specific efficiency-index, $e_i = e_i(t, Y)$.

Starting out with a conventional production function without efficiency indexes, $Y = F(X_1, \ldots, X_n)$, this yields the long-run factor demand functions:

$X_i^* = X_i^*(Y, P_1, \dots, P_n)$, $i = 1, \dots, n$, and the long-run cost function $C^* = C^*(Y, P_1, \dots, P_n)$. Using the shadow price result, short-run factor demands can be directly derived from long-run factor demands (see Section 3). If it is assumed that only X_1 is quasi-fixed, the short-run factor demands will be of the form $X_i = X_i(Y, X_1, P_2, \dots, P_n)$, $i = 2, \dots n$. If we now assign an efficiency index, e_i, to each factor, this yields a production function with disembodied factor-augmenting efficiency indexes, $Y = F(e_1 X_1, \dots, e_n X_n)$. The functional form, $F(\cdot)$, is the same, and by rewriting the costs as $C = P_1 X_1 + \cdots + P_n X_n = (P_1/e_1)\cdot(e_1 X_1) + \cdots + (P_n/e_n)\cdot(e_n X_n)$, it is easy to prove that the following long-run factor demands result:

$$X_i^* = \frac{1}{e_i} X_i^* \left(Y, \frac{P_1}{e_1}, \dots, \frac{P_n}{e_n} \right), i = 1, \dots, n. \tag{35}$$

And the following long-run cost function:

$$C^* = C^* \left(Y, \frac{P_1}{e_1}, \dots, \frac{P_n}{e_n} \right). \tag{36}$$

From (35) it is seen that the *efficiency-corrected* factor levels, $e_i X_i^*$, respond to the *efficiency-corrected* factor prices, P_i/e_i. The mathematical functions $X_i^*(\cdot)$ and $C^*(\cdot)$ are the same as without efficiency indexes, so the point is that it is quite easy to introduce disembodied factor-augmenting technological progress (or effects from other exogenous factors) and/or scale effects into any system of long-run factor demand functions (or into any long-run cost function). Similarly, short-run factor demands are given as (assuming here, that there is only one quasi-fixed factor, X_1):

$$X_i = \frac{1}{e_i} X_i \left(Y, e_1 X_1, \frac{P_2}{e_2}, \dots, \frac{P_n}{e_n} \right), \quad i = 2, \dots, n. \tag{37}$$

Generally, one can deduce all concepts from a "stripped down" constant returns to scale cost or production function without technological change/other exogenous variables, and afterwards introduce exogenous factors and scale effects via the efficiency indexes (cf. Section 1). The indexes are introduced by multiplying all factor levels and dividing all factor prices with the corresponding efficiency indexes, just as if the factors and factor prices were pre-corrected for efficiency.[9]

[9] The only exception to this rule is the marginal costs, where it must be remembered that the efficiency indexes themselves can be functions of Y. Thus, if the efficiency indexes are Y-dependent, the expression yielding long-run marginal costs, $MC^* \equiv \partial C^*/\partial Y$, must be recalculated in the light of this. This being done, short-run marginal costs can still be found by inserting the shadow price(s) of the quasi-fixed factor(s) into the long-run marginal cost function.

The indexes themselves could be formulated as, e.g.

$$\log(e_i) = \omega_{i1} t + 0.5\,\omega_2 t^2 + \psi_{i1} \log(Y) + 0.5\,\psi_2 \log^2(Y) + \phi t \log(Y),$$
$$i = 1, \ldots, n. \tag{38}$$

In this formulation, the second-order effects and the cross effect between t and $\log(Y)$ are assumed to be identical in each of the n efficiency indexes. In (38), the expression $\omega_{i1} + \omega_2 t + \phi \log(Y)$ indicates – multiplied by 100 – by how many per cent the efficiency of factor i increases from period t to period $t+1$, and the expression $\psi_{i1} + \psi_2 \log(Y) + \phi t$ denotes by how many per cent the efficiency index increases, if the production level is increased by 1%. Other formulations might prove equally useful, and other exogenous factors in addition to time, t – such as machine efficiency, squared time differences in the capital stock (representing internal costs of installing/removing capital equipment), mean age of the capital stock (capturing vintage-effects), eduction level/human capital, fuel-efficiency, climate, land, infrastructure, public R&D, etc. – may enter the efficiency indexes as well.

Fortunately, it turns out that efficiency indexes of the form (38) render any stripped down flexible cost function flexible in t and Y, as shown below (for more details and a numerical example, the reader is referred to Thomsen, 1998, Appendix B). Consider a stripped down long-run translog cost function, $\log(C^*) = a_0 + \log(Y) + \boldsymbol{A}' \log(\boldsymbol{P}) + 0.5 \log(\boldsymbol{P}')\boldsymbol{B} \log(\boldsymbol{P})$ where $\boldsymbol{A}$ is a $n \times 1$ vector of parameters summing to unity, and $\boldsymbol{B}$ is a $n \times n$ symmetric matrix of parameters with rows and columns summing to zero. Consider also the standard long-run translog cost function: $\log(C^*) = a_0 + \boldsymbol{A}' \log(\boldsymbol{P}) + a_t t + a_y \log(Y) + 0.5 \log(\boldsymbol{P}')\boldsymbol{B} \log(\boldsymbol{P}) + \boldsymbol{B}_t' \log(\boldsymbol{P})t + \boldsymbol{B}_y' \log(\boldsymbol{P}) \log(Y) + 0.5 a_{tt} t^2 + b_{ty} t \log(Y) + 0.5 a_{yy} \log^2(Y)$, where a_t, a_y, a_{tt}, b_{ty} and a_{yy} are scalars, and $\boldsymbol{B}_t$ and $\boldsymbol{B}_y$ are $n \times 1$ vectors of parameters summing to zero (see Christensen et al., 1971,1973, or Diewert and Wales, 1987, p. 46). Now, let $\boldsymbol{\Omega}_1, \boldsymbol{\Omega}_2, \boldsymbol{\Psi}_1, \boldsymbol{\Psi}_2$ and $\boldsymbol{\Phi}$ be $n \times 1$ vectors of the efficiency parameters of (38) (i.e. with identical elements of the three vectors $\boldsymbol{\Omega}_2, \boldsymbol{\Psi}_2$ and $\boldsymbol{\Phi}$). It can be shown (see Thomsen, 1998, Appendix B, for more details) that augmenting the stripped down translog with efficiency indexes of the form (38) results in the standard translog, with the following relationships between the $2n+3$ free parameters of $a_t, a_y, a_{tt}, b_{ty}, a_{yy}, \boldsymbol{B}_t$ and $\boldsymbol{B}_y$ on the one hand, and the $2n+3$ free parameters of $\boldsymbol{\Omega}_1, \boldsymbol{\Omega}_2, \boldsymbol{\Psi}_1, \boldsymbol{\Psi}_2$ and $\boldsymbol{\Phi}$ on the other: $a_t = -\,\boldsymbol{\Omega}_1'\boldsymbol{A}$, $a_y = 1 - \boldsymbol{\Psi}_1'\boldsymbol{A}$, $\boldsymbol{B}_t = -\,\boldsymbol{B}\boldsymbol{\Omega}_1$, $\boldsymbol{B}_y = -\,\boldsymbol{B}\boldsymbol{\Psi}_1$, $a_{tt} = -\,\boldsymbol{\Omega}_2'\boldsymbol{A} + \boldsymbol{\Omega}_1'\boldsymbol{B}\boldsymbol{\Omega}_1$, $a_{yy} = -\,\boldsymbol{\Psi}_2'\boldsymbol{A} + \boldsymbol{\Psi}_1'\boldsymbol{B}\boldsymbol{\Psi}_1$, $b_{ty} = -\,\boldsymbol{\Phi}'\boldsymbol{A} + \boldsymbol{\Psi}_1'\boldsymbol{B}\Omega_1$. These relationships yield a completely new way of interpreting the translog trend- and scale-parameters, since these can be directly translated into efficiency parameters, and vice versa. And since the stripped down translog can mimic the factor price substitution of any other stripped down flexible cost function to a second-order degree, it follows that efficiency indexes of the form (38) render any stripped down flexible cost function fully flexible in t and Y.

In spite of (38) being fully flexible, one could, however, relax the restriction that the second-order effects are identical for the different indexes. Thus, a more general formulation would be the following:

$$\log(e_i) = \omega_{i1} t + 0.5\,\omega_{i2} t^2 + \psi_{i1} \log(Y) + 0.5\,\psi_{i2} \log^2(Y) + \phi_i t \log(Y),$$
$$i = 1, \ldots, n. \tag{39}$$

Here, if $\omega_{i1} = \omega_1, \omega_{i2} = \omega_2$, and $\phi_i = \phi$ (i.e., the ω_{i1}'s the ω_{i2}'s, and the ϕ_i's are identical), technological change is Hicks-neutral (unbiased). If $\omega_{i1} = \omega_{i2} = \phi_i = 0$, except the ω_{i1}-, ω_{i2}-, and ϕ_i-parameters of the labour efficiency index, technological change is Harrod neutral (labour augmenting). And if $\omega_{i1} = \omega_{i2} = \phi_i = 0$, except the ω_{i1}-, ω_{i2}-, and ϕ_i-parameters of the capital efficiency index, technological change is Solow neutral (capital augmenting).

If $\psi_{i1} = \psi_1, \psi_{i2} = \psi_2$, and $\phi_i = \phi$, the production function is homothetic (unbiased scale effects). Specifically, if $\psi_{i1} = \psi_1, \psi_{i2} = 0$, and $\phi_i = 0$, the production function is homogenous of degree $1/(1-\psi_1)$. The restriction $\psi_{i1} = \psi_{i2} = \phi_i = 0$ implies constant returns to scale.[10]

Regarding the efficiency index approach, it is finally worth mentioning that the way these efficiency indexes influence the long-run demands can be decomposed using the following simple relationship:

$$\partial \log(\boldsymbol{X}^*) = -(\boldsymbol{I} + \boldsymbol{E})\, \partial \log(\boldsymbol{e}), \tag{40}$$

where $\boldsymbol{X}^*$ is a $n \times 1$ vector of the long-run factor levels, $\boldsymbol{I}$ is a $n \times n$ identity matrix, $\boldsymbol{E}$ is a $n \times n$ matrix of long-run partial price elasticities, and $\boldsymbol{e}$ is a $n \times 1$ vector of efficiency indexes. From this relationship, it is seen that if there is no factor substitution ($\boldsymbol{E} = \boldsymbol{0}$), an increase in the efficiency of factor i by 1% simply causes a corresponding decrease in the use of factor i itself by 1%. If there is non-zero factor substitution, the use of factor i would fall by *less* than 1%, and this is "used" to reduce the levels of one or more of the other factors as well.[11] If the formulation (39) is used, the trend- and scale-effects can be decomposed into

$$\partial \log(\boldsymbol{X}^*)/\partial t = -(\boldsymbol{I} + \boldsymbol{E})(\boldsymbol{\Omega}_1 + \boldsymbol{\Omega}_2 t + \boldsymbol{\Phi} \log(Y)), \text{ and}$$
$$\partial \log(\boldsymbol{X}^*)/\partial \log(Y) = -(\boldsymbol{I} + \boldsymbol{E})(\boldsymbol{\Psi}_1 + \boldsymbol{\Psi}_2 \log(Y) + \boldsymbol{\Phi} t) + \boldsymbol{i},$$

where $\boldsymbol{\Omega}_1, \boldsymbol{\Omega}_2, \boldsymbol{\Psi}_1, \boldsymbol{\Psi}_2$ and $\boldsymbol{\Phi}$ are $n \times 1$ vectors of the efficiency parameters of (39), and $\boldsymbol{i}$ is a vector of ones.

---

[10] Regarding scale-effects, the quadratic formulation of the Y-effects in (39) (or in the more restricted (38)) is flexible enough to make possible the presence of U-shaped long-run average costs, $AC^* = C^*/Y$. With $\psi_{i1} = \psi_1, \psi_{i2} = \psi_2$, and $\phi_i = \phi$, the scale-effects are of the so-called "Nerlove-Ringstad" type (cf. Zellner and Ryu, 1998).

[11] This is the normal case. However, if the substitution is very large, the use of factor i itself might even rise, if it gets more efficient. This would be the case if the own-price elasticity of factor i is below -1. Besides, a rise in the efficiency of factor i *rises* the use of those of the other factors that are complementary to i (negative cross-price elasticities).

To summarize this section, the advantage of the efficiency index approach lies in two points. Firstly, it is easy to introduce these indexes – and new exogenous variables inside them – into any stripped down no technological change and constant returns to scale cost function or factor demand system. Secondly, the interpretation of the parameters of such efficiency indexes is much more straightforward – and comparable over different cost functions – than, e.g. trying to figure out the interpretation of a trend in a cost share or factor intensity (for instance, the trend-parameters of the efficiency indexes can be directly linked to the textbook concepts of Hicks-, Harrod- and Solow neutrality).

7. A simple example using the Berndt–Wood data set

To illustrate the above techniques, an illustrative estimation on the much used Berndt–Wood data set (covering U.S. manufacturing over the period 1947–1971) is presented. In the estimation, capital (K), labour (L), energy (E) and materials (M) are described as a function of the four corresponding factor prices, the production level (Y), and time (t).[12]

Diewert and Wales (1987) use the same data set to estimate – among other things – translog- and generalized Leontief cost functions, with full flexibility regarding price elasticities, trend- and scale-effects. As I see it, however, the scale effects of these estimations are not fully convincing, because t and $\log(Y)$ are highly correlated in the data set (with a correlation coefficient of 0.971). In order to avoid these unpleasant multicollinearity problems and obtain more robust results, I assume constant returns to scale. Hence, I use a stripped down long-run generalized Leontief cost function extended with efficiency indexes (39) – the latter without Y-effects, but with unrestricted t^2-terms; i.e. two trend parameters in each of the four efficiency indexes.

Abstracting from possible cross-effects in the adjustment of K and L, it is assumed that K and L adjust to their long-run levels according to the following simple error correction mechanism (here for K):[13]

$$\Delta\log(K_t) = \lambda_1 \Delta\log(K_t^*) + \lambda_2\, [\log(K_{t-1}^*) - \log(K_{t-1})] + u_t, \tag{41}$$

[12] See Berndt and Wood (1975) and Berndt and Khaled (1979) regarding the construction of the data.

[13] This paper does not deal with the problem of explaining *why* the quasi-fixed production factors are not flexible – and precisely *how* the quasi-fixed factors are expected to adjust over time. That would imply discussing adjustment costs and uncertainty, and going into that discussion would be beyond the scope of this paper. The interested reader is instead referred to Nickell (1985), Galeotti (1996), or Atkinson and Halvorsen (1998). Nickell demonstrates that under some reasonable assumptions regarding the short-run cost function, adjustment costs, and expectation rules, dynamic optimization implies that the quasi-fixed factors adjust to their long-run levels by means of simple error correction mechanisms.

Table 2
Estimation of a factor demand system derived from a long-run generalized Leontief cost function, with K and L quasi-fixed and E and M flexible

	Long-run partial price elasticities				% Growth of eff. ind.		Adjustment				
	P_K	P_L	P_E	P_M	1949	1971	λ_1	λ_2	%SEE	DW	JB
K	− 0.49	0.50	− 0.34	0.33	12.4	− 13.9	0.10	0.25	2.2	0.73	4.1
	(0.06)	(0.12)	(0.20)	(0.14)	(2.1)	(2.3)	(0.05)	(0.04)			
L	0.11	− 0.31	0.04	0.17	− 0.6	3.5	0.78	1.03	1.8	0.65	1.2
	(0.03)	(0.11)	(0.02)	(0.09)	(0.8)	(0.7)	(0.03)	(0.07)			
E	− 0.34	0.16	− 0.41	0.59	12.6	− 15.8	•	•	2.3	0.91	1.4
	(0.19)	(0.07)	(0.09)	(0.26)	(2.8)	(3.3)					
M	0.04	0.09	0.07	− 0.19	− 1.2	2.6	•	•	2.0	1.00	0.8
	(0.02)	(0.05)	(0.03)	(0.07)	(0.5)	(0.6)					

Note: The estimation period is 1948–1971, and the price elasticities are of the type $\partial \log(X_i)/\partial \log(P_j)$. Standard errors are in parentheses. JB is the Jarque/Bera test for normality of the residuals and should not exceed $\chi^2_{5\%}(2) = 6.0$. Log likelihood = 265.66. The cost function is concave at all data points.

where λ_1 and λ_2 are adjustment parameters. Utilizing the results of Sections 5 and 6 to obtain short-run factor demand equations for E and M, the result shown in Table 2 is obtained.[14]

The 4×4 numbers to the left are long-run partial price elasticities (evaluated in 1971), where, inter alia, it is seen that K and E are complementary. The two columns of growth rates of the efficiency indexes (in 1949 and 1971, respectively) show, e.g. that the efficiency growth of K changed from 12.4% p.a. in 1949 to − 13.9% p.a. in 1971. The standard error (SEE) of the residuals is in the range 1.8–2.3%, and the Jarque–Bera tests do not indicate serious problems with the assumption of the error terms being normally distributed, whereas the Durbin–Watson tests indicate the presence of autocorrelated error terms. However, it must be kept in mind that the dynamic formulation is very simple, with a total of only four adjustment parameters.[15]

The growth rates of the K- and E-efficiencies have been decreasing over the estimation period, hinting that technological progress and capital/energy could be conceived of as being complementary in the second half of the estimation period. In Fig. 3, the historical fit is depicted.

In the figure, among other things it can be seen that K^* exceeds K in most of the estimation period (the average gap being 17%), which is reflected in E^* also exceeding E (due to the complementarity of K and E). If the production level, Y, is increased by 1% in the dynamic model, the adjustment looks as seen in Fig. 4.

In the long run, all factors are also increased by 1%, due to the imposed constant returns to scale. However, in the short run K and L react sluggishly,

[14] More precisely, Eq. (22) with $h(Y) = Y$ is used as regards K^* and L^*, and Eq. (27) with $h(Y) = Y$ is used as regards E and M. All of these four demand equations are extended with efficiency indexes in the manner shown in (35) and (37), respectively, and logarithms are taken on both sides of the equality signs in the E- and M-equations. The efficiency indexes are of the form (39) with the last three terms suppressed. The estimation procedure is maximum likelihood, assuming that the disturbance terms are serially uncorrelated and follow a multivariate normal distribution with zero means and constant variances and covariances.

[15] The presence of autocorrelation indicates that the model is probably somewhat mis-specified. Potential autocorrelation-causing left-out variables could be the "true" efficiency indexes (imitated in the estimation by quadratic trends) – i.e., for instance, machine efficiency for K, education level/human capital for L, and energy efficiency for E. Even abstracting from such left-out variables, the presence of lagged endogenous variables in the K- and L-equations renders the ML estimates inconsistent, if the error terms are autocorrelated. A relatively simple remedy against this could be to formulate a simple AR(1) process in the four error terms (i.e., $u_t = \rho u_{t-1} + \varepsilon_t$), but since the estimation is for illustrative purposes only, that would be beyond the scope of this paper. In a more ambitious empirical analysis, IM-tests (Information Matrix) could be carried out, to test whether the model is mis-specified. Also, since the sample contains 24 observations only, bootstrapping techniques could be used to render more reliable small-sample distributions of the parameters. See e.g. Davidson and MacKinnon (1993) for details.

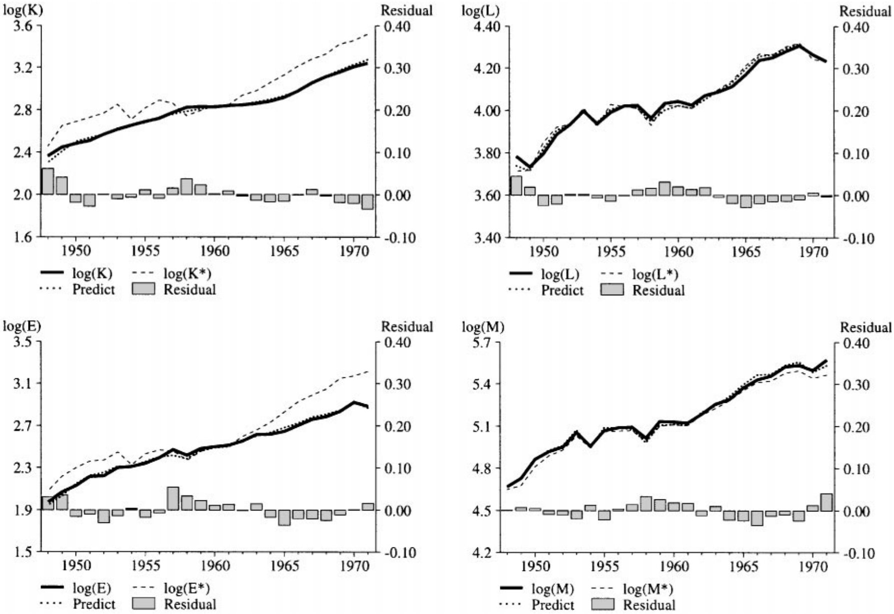

Fig. 3. Historical fit.

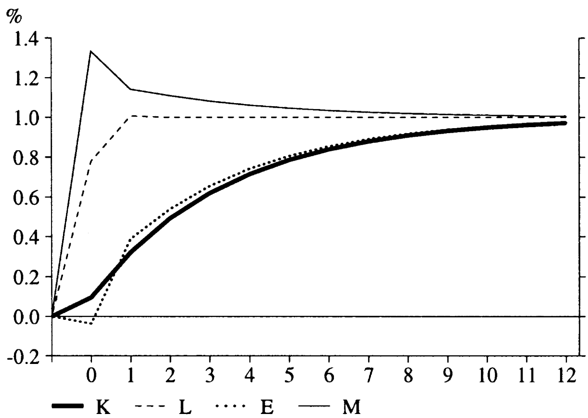

Fig. 4. The dynamic effect of a 1% increase in the production level.

Table 3
Effects of technological change on inputs and total costs

	K	L	E	M	C
1948	− 2.0	− 1.1	− 1.7	− 0.1	− 0.6
1971	− 0.8	− 0.8	+ 2.6	− 0.8	− 0.6

Note: Table entries are $100\,\partial \log(X_i)/\partial t$ for input i and $100\,\partial \log(C)/\partial t$ for total costs.

causing M to overshoot, whereas E follows K quite closely due to the complementarity of K and E. The sluggishness of L suggests that the firms perform labour-hoarding in the first year of the adjustment process.[16]

Keeping in mind the simplicity of the dynamic adjustment, the model seems reasonable, perhaps apart from the fact that the growth rate of the efficiency indexes of K and E changes rather rapidly. The effects of the efficiencies on factor demand are, however, propagated through the matrix of price elasticities (cf. Eq. (40) in Section 6), and the effects on factor demand and costs are much less different, as shown in Table 3.

From Table 3, it can be seen that the annual decrease of total costs due to technological progress has been 0.6% in both 1948 and 1971.

8. Conclusions

In this paper, it has been shown that it is possible to start out with a no technical progress and constant returns to scale ("stripped down") long-run cost function, adding dynamics, trend- and scale-effects by means of shadow prices and efficiency indexes.

For instance, using the original (Diewert) long-run generalized Leontief cost function (without trend- and scale-effects) as a starting point, one can analytically compute all the usually employed short-run concepts by means of shadow prices (see Sections 2.2 and 5), and trend- and scale-effects can be added easily and unambiguously by means of the efficiency index approach of Section 6. This makes the long-run generalized Leontief cost function a promising candidate for dynamic factor demand modelling, and Section 7 shows that it is possible to estimate a quite plausible KLEM factor demand system (of which K and L are

[16] Using the same model, I have obtained quite similar results (K and E also being complementary, and L performing labour-hoarding) on Danish aggregate data for K, L E and M over the period 1957–1989. The interested reader is referred to Thomsen (1995).

assumed quasi-fixed) on the Berndt–Wood data set, using this cost function as a starting point, and extending it by means of shadow prices and efficiency indexes.

All in all, the paper shows that there is no need to spend time and effort on inventing new short-run cost functions, including figuring out how to best introduce trend- and scale-effects (or effects from other exogenous factors) into these. The effort should instead be concentrated on finding a promising stripped down long-run cost function, $C^*(Y, P_1, \dots, P_n) = Y \cdot c^*(P_1, \dots, P_n)$, letting the shadow prices and efficiency indexes take care of the rest.

Acknowledgements

I would like to thank my colleagues, especially Jes Asger Olsen, for useful comments on this paper.

Appendix A. Approximations between short- and long-run factor demand

See Thomsen (1998), Section 7 for more details on this. The idea is to use a logarithmic linearization of the relationship between the factors and the factor prices around the long-run levels of the former, $\boldsymbol{X}^*$:

$$\partial \log(\boldsymbol{X}^*) = \boldsymbol{E}\, \partial \log(\boldsymbol{P}). \tag{A.1}$$

Here, $\boldsymbol{X}^*$ is a $n \times 1$ vector of the n factors, $\boldsymbol{E}$ is a $n \times n$ matrix of long-run partial price elasticities and $\boldsymbol{P}$ is a $n \times 1$ vector of factor prices. Partitioning (A.1) into the k quasi-fixed and $l = n - k$ flexible factors, the following is obtained:

$$\partial \log \begin{bmatrix} \boldsymbol{X}_k^* \\ \boldsymbol{X}_l^* \end{bmatrix} = \begin{bmatrix} \boldsymbol{E}_{kk} & \boldsymbol{E}_{kl} \\ \boldsymbol{E}_{lk} & \boldsymbol{E}_{ll} \end{bmatrix} \partial \log \begin{bmatrix} \boldsymbol{P}_k \\ \boldsymbol{P}_l \end{bmatrix}. \tag{A.2}$$

From (A.2) it follows that the virtual/shadow prices, called $\tilde{\boldsymbol{P}}_K$, of the quasi-fixed factors can be approximated as (please note in the two following formulas that the shadow price method leaves $\boldsymbol{P}_l$ unaltered):

$$\log(\tilde{\boldsymbol{P}}_k) - \log(\boldsymbol{P}_k) \approx \boldsymbol{E}_{kk}^{-1}\, [\log(\boldsymbol{X}_k) - \log(\boldsymbol{X}_k^*)], \tag{A.3}$$

provided that the submatrix $\boldsymbol{E}_{kk}$ is non-singular. Inserting (A.3) into $\boldsymbol{X}_l^*$ in (A.2) results in

$$\log(\boldsymbol{X}_l) \approx \log(\boldsymbol{X}_l^*) + \boldsymbol{E}_{lk}\boldsymbol{E}_{kk}^{-1}\, [\log(\boldsymbol{X}_k) - \log(\boldsymbol{X}_k^*)]. \tag{A.4}$$

These are *approximated* short-run factor demands for the flexible factors ($\boldsymbol{X}_l$), but it must be made clear, that the approximation may be less satisfactory far from the long-run levels. But under normal circumstances – i.e. provided that the quasi-fixed factors do not deviate too much from their long-run levels – the formula could prove very useful.

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